Multiscale dynamics of a Foucault pendulum for gravitational measurements

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Abstract. This paper summarises some ongoing experimental research in frame-dragging measurement in which the dynamics of the experiment operate on three distinctly different time scales. The experiment is designed to detect the relativistic effect of frame-dragging on a small test mass, measured on Earth using an instrumented Foucault pendulum. Gravitoelectromagnetism (GEM) has been used to develop a simple but sufficiently accurate theoretical model of the frame-dragging effect caused by the massive rotating body of the Earth influencing the motion of the bob of a laboratory-scale Foucault pendulum located in the city of Glasgow. In the experiment the operational time scales relate to the swing of the pendulum which has a natural time constant of a few seconds, the cyclical precession of the pendulum for which the time constant is typically of the order of a day, dependent on location, and finally the many months required to build up the minute frame-dragging signal to a potentially measurable level. The experiment requires careful separation of the signal from the complex noise floor, to enable the numerical calculation of the signal from the measured data, in relation to the fixed inertial frame associated with a suitable guide star and also to the terrestrial location of the laboratory. The measurement must also account for the relatively large effect on the measurement of small natural fluctuations in the acceleration due to gravity at the measurement location. This experiment has been under continuous development since its conception in 2017 and the build and test phases are nearing completion. There have been several important materials considerations, notably in the selection of the pendulum fibre and bob material, and also relating to the electromagnetic performance of the drive system and the instrumentation, all contributing to the reliable continuous operation of the pendulum over time. In this paper these practical issues are summarised, and the roles of the time-scales of this experiment are also discussed.

Keywords: Frame dragging, General Relativity, Foucault pendulum.

1 Modelling a terrestrial Foucault pendulum

1.1 Equations of motion

A mathematical model for the terrestrial Foucault pendulum was derived in [1, 2] and this forms the basis for simulations to examine its behaviour both at the chosen site of Glasgow, Scotland, and also at the key geographical locations of the poles, the equator, and 45° north and south. The model is defined in terms of two terrestrial Cartesian coordinates, x and y. Generalised forces Q_x and Q_y are introduced so that appropriate excitation terms can be included in the model to simulate a physical drive system to enable continuous motion of the pendulum over time [3]. Quadratic aerodynamic damping has also been included, numerically scaled by means of the constant η. This aerodynamic damping quantity is defined as $\eta = \frac{\rho_{air} C_D \pi r_{bob}^2}{2 m_{bob}}$ (see the caption to Fig. 3). Physically realistic values for η are calculated in the numerical modelling work reported initially in [1] and then further in [3], see the caption to Fig. 3. The physical pendulum is shown schematically in Figs. 1 and 2, in both the Earth-fixed frame E and the laboratory-fixed frame P respectively.

$$\ddot{x} + \eta|\dot{x}|\dot{x} - 2\dot{y}\varOmega sin\phi - x\varOmega^2 + \frac{gx}{l\sqrt{1-\frac{x^2+y^2}{l^2}}} = Q_x \tag{1}$$

$$\ddot{y} + \eta|\dot{y}|\dot{y} + 2\dot{x}\varOmega sin\phi - y\varOmega^2 sin^2\phi + r\varOmega^2 sin\phi cos\phi + \frac{gy}{l\sqrt{1-\frac{x^2+y^2}{l^2}}} = Q_y \tag{2}$$

Latitude ϕ is defined on plane EYZ in the Earth-centred frame EXYZ. Plane EYZ is coplanar with the local ground-located laboratory frame pyz. The rotation rate of the Earth is given by $\varOmega$. The Earth radius at the location of the pendulum is defined by r, the length of the pendulum is denoted by l, and the nominal local value of the acceleration due to gravity is stated here as g. The angular position of the swinging pendulum is directly obtained from x and y, and is defined in Figs. 1 and 2 by α.

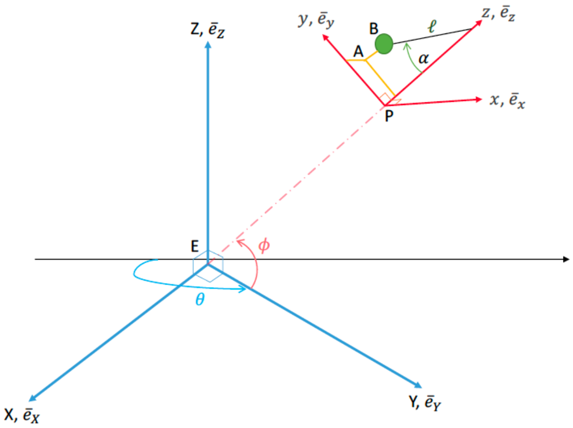


Fig. 1. Earth-fixed frame $EXYZ$ and laboratory-fixed frame $pxyz$, [1].

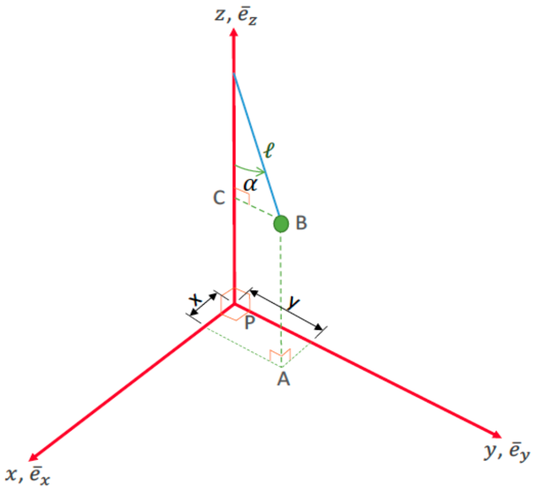
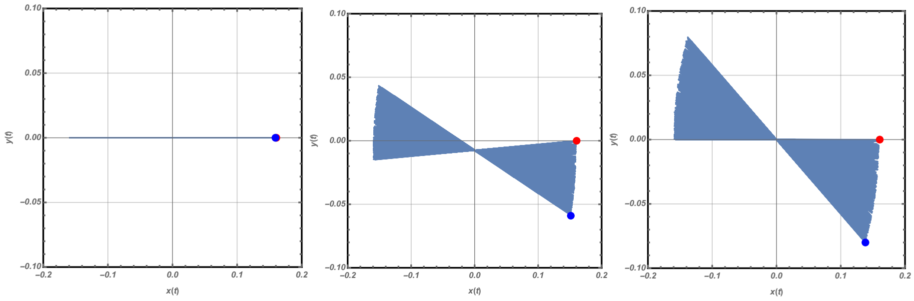


Fig. 2. Laboratory frame $pxyz$, [1].

There is a location dependent DC term in equation (2) that represents a component of the centrifugal force due to the Earth's rotation. This is of the form: $\mathrm{r}\Omega^2\sin\phi\cos\phi$ which can be written as $\frac{1}{2}\mathrm{r}\Omega^2 \sin 2\phi$. This term is zero at the equator where $\phi = 0°$, and is also zero at the North Pole where $\phi = 90°$. The DC offset effect of this term is a maximum at locations where $\phi = 45°$. Equations (1) and (2) may be directly integrated in *Mathematica*™ using the built-in NDSolve integrator package. The results of numerical integrations over 2 hours are given in Fig. 3, for the test data in the caption.



(a) (b) (c)

Fig. 3. Results of numerical integration of equations (1) and (2) in *pxyz* Cartesian coordinates. (a) Equator $\phi = 0$, (b) $\phi = 45°$ north, (c) North Pole $\phi = 90°$. $l = 4.359 \pm 0.001$ m, $m_{bob} = 2.52529$ kg, $r_{bob} = 0.047$ m, $r = 6363.18 \times 10^3$m (Glasgow), $\Omega = 7.2921150 \times 10^{-5}$ rad/s, $g = 9.8156$ m/s^2 (nominal at Glasgow), $t_{end} = 7200$ s, $\rho_{air} = 1.189$ kg/m^3, $C_D = 10^{-6}$, $x_0 = 0.16$ m, $y_0 = 0$ m, $\dot{x}_0 = \dot{y}_0 = 0$ m/s, red dot denotes (x_0, y_0), blue dote denotes (x_{end}, y_{end}).

There is no Foucault precession at the equator (Fig. 3(a)), some precession but with maximum DC offset, at 45° north or south (Fig. 3(b)), and the maximum precession possible is seen to occur at the north or south poles (Fig. 3(c)). The laboratory in Glasgow is located between 45° north and the north pole, at 55.862°. In the southern hemisphere the DC offset and the precession are both in the opposite direction. In the northern location of the laboratory the pendulum is observed to precess clockwise with respect to the laboratory. This indicates that the Earth is rotating anticlockwise with respect to the inertial frame. Note that Q_x and Q_y model rectangular wave functions that are scaled in amplitude and phased in time to represent the electromagnetic force components from a closed loop drive system installed in the laboratory. This drive maintains continuous pendulum motion against friction and aerodynamic damping [3].

1.2 Experimental system design

All data used in the simulations from the numerical integration of equations (1) and (2) has been obtained from fundamental measurements of the experimental pendulum in the laboratory or taken directly from published or measured geographical data for the laboratory location in Glasgow, Scotland, UK. The pendulum is driven by a closed loop electromagnetic drive [3,4,5] based around a pair of concentric sense and drive coils located directly underneath the equilibrium position of the bob. A neodymium magnet is fitted within a recess in the base of the bob [5] and the electromagnetic interaction between the (outer) sense coil and the magnet as the bob swings over the outer sense coil generates a voltage pulse in the sense coil which is then used to initiate a timed response pulse in the inner drive coil to provide a calibrated push to the bob as it carries on through its swing. The sense and drive coils were wound professionally by Custom Transformers Ltd of Malmesbury, England, UK, to the author's specific design. This was necessary to ensure that the turns and layers in each coil were laid down perfectly uniformly. This type of drive system is analysed in detail in [3,4], and is shown in those papers to be both capable of maintaining continuous motion of the pendulum and of minimising the undesirable ellipticity precession that is inherent in all Foucault pendulums, and which increases systematically with reduced pendulum length proportional to $l^{-5/2}$. The pendulum fibre is made of a commercial monofilament leader line, marketed as *KastKing Durablend*™, of nominal diameter 1.5 mm and with a claimed breaking strength of 890 N [5]. There are some key features of the current laboratory pendulum which warrant some summary discussion here. The upper pivot fixture is a reworked version of the mechanically decoupled system of [5]. The re-work was required due to an observed inconsistency in the behaviour of the bearing over time, and now the upper fixture is a rigid design in which the clamped fibre emerges from an almost perfectly circular hole within the lower half of the fibre attachment sub-assembly, as shown in Fig. 4(a). A resin-potted collar is used to fix the fibre-end securely within the attachment sub-assembly. At the other end of the fibre, where it enters the bob, further modifications have been carried out to remove all ferrous material completely from the bob in order not to exacerbate any tiny electromagnet asymmetries in the system. The three principal mass moments of inertia of the bob have also been slightly modified from the design of [5] to ensure that the natural 'wobble' frequencies about the three local axes are far from the pendulum swing and precession frequencies. The magnetic flux density distribution of the neodymium magnet recessed into the bob underside has also received considerable attention since originally reported in [5]. This was necessary because of manufacturing inaccuracies in the sintering process that were found to lead to inconsistent magnetic flux densities in commercially available magnets when measured circumferentially at a constant radial position. In this further experiment ten magnets were investigated and only two were found to vary within an acceptably low range of 65 Gauss when measured around the circumference of the magnet at a position 2 mm from the edge, i.e. to within 10% of the mean value. The other eight magnets performed less well, with a circumferential variation of up to 191 Gauss in the worst case. The best magnet from this random selection of ten showed a variation of just 61 Gauss against a mean of 1003 Gauss, when measured this way (a deviation of 6%) Fig. 5(a). This magnet was subsequently fitted into the bob, which is shown instantaneously in its swing at a position just past the coil pack, in Fig 4(b).

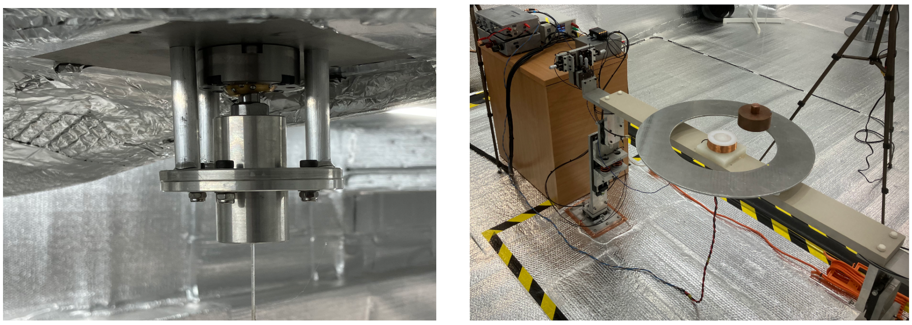
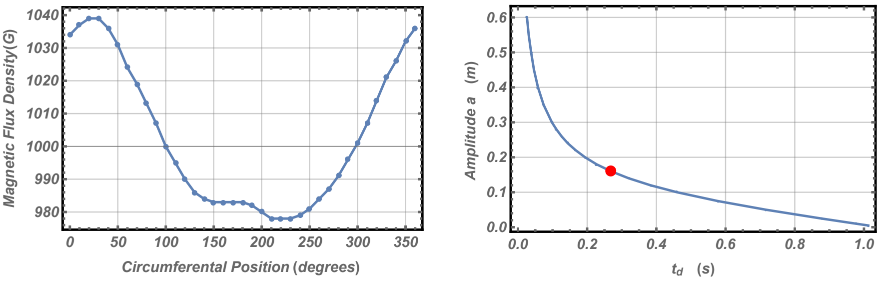

(a) (b)

Fig. 4. (a) Upper fixture in laboratory roof showing the pendulum fibre emerging from the lower half of the fibre attachment sub-assembly, (b) Bob having passed over the concentric sense and drive coils and passing over the annulus of the Eddy current damper ring, on its outward swing.



(a) (b)

Fig. 5. (a) Magnetic flux density (Gauss) against circumferential position (degrees) of the best magnet from the sample of 10, showing a peak-to-peak variation of 61 Gauss over the circumference of the magnet, (b) Electromagnetic drive characteristics [3]: pendulum swing peak amplitude (m) as a function of delay time (s), red dot defines the chosen operating point where delay time of $t_d = 269$ ms gives a practically convenient peak pendulum amplitude (major axis) of $a = 16$ cm.

The electromagnetic pusher excitation has been modelled [3] such that the peak swing amplitude is a function of the delay time between receipt of the detection pulse in the outer sense coil and the drive pulse in the inner drive coil. This functionality has been built into the electronic hardware fitted in the boxed units shown at the upper left of the photograph in Fig. 5(b). The updated experimental system has demonstrated Foucault precession faultlessly over five weeks of continuous testing at the time of writing (21/05/2024). The expected Foucault precession is given by $360 \sin \phi$ which gives 297.968 °/sidereal day in Glasgow (55.862° north, 4.245° west), which equates to 12.449 °/hour. This compares with a prediction of 12.487 °/hour from the numerical integration of equations (1) and (2). On test the experiment generates a Foucault precession of 180° over 14 h 27m 30s, which converts to 12.449(57) °/hour. So, the experimentally measured value is identical to the predicted value to 3 decimal places for the location of Glasgow. The value extrapolated from the numerical integrations over-predicts by 0.3%. A summary of the experimental data is given next.

1.3 Experimental data

The length of the pendulum is defined from the point of emergence from the lower half of the fibre attachment sub-assembly (Fig. 4(a)) to the location of the centre of mass of the bob. A laser range finder with a stated accuracy of $\pm$ 1 mm per 5 m was used to obtain $l = 4.359 \ \pm 0.001$ m. The gap between the neodymium magnet and the top face of the concentric coil pack is initially arbitrary, within practical operating limits, but once set must be maintained to ensure consistent drive characteristics. This gap was set using a slip gauge to $l_{gap} = 6 \pm 0.1$ mm. The set-up of the drive system [3] was based on this l_{gap} value and so the characteristic of Fig. 5(b) specifically reflects this setting. The bob mass, density, and radius are, respectively, $m = 2.525(29)$ kg, $\rho_{cu} = 8940$ kg/m^3, and $r_{bob} = 0.047$ m. The logarithmic decrement for the swinging motion was taken over 40 full cycles giving a damping ratio of $\xi = 0.000185$, and from this a quality factor of $Q = 1/2\xi = 2703$. The period of free damped vibration (in swing) was measured as $T_d = 4.1872$ s, and the calculated value for the period of free undamped vibration was $T_u = 4.1876$ s. These values led to the following natural frequencies, respectively, of $\omega_d = 1.50059$ rad/s, and $\omega_u = 1.50060$ rad/s. The nominal value of the local acceleration due to gravity in Glasgow is $g_{local} = 9.8156$ m/s^2 and the measured peak fluctuation in the local acceleration due to gravity is ± 100 μGal, or $\pm 10^{-6}$ m/s^2 [3]. Therefore, the upper value of local acceleration due to gravity in Glasgow is $g_{local_U} = 9.815601$ m/s^2 and the lower value of local acceleration due to gravity is $g_{local_L} = 9.815599$ m/s^2. Calculation of the frame-dragging effect of Lense-Thirring precession has been discussed in some detail in previous publications and from a simplified GEM analysis this is predicted to be $\Omega_{LT} = 162.6$ mas/year [5] for $G = 6.67408 \times 10^{-11}$ m^3 kg^{-1}s^{-2}. $M_\oplus = 5.972 \times 10^{24}$ kg, $\Omega_\oplus = 7.2921150 \times 10^{-5}$ rad/s, $c = 2.99792488 \times 10^8$ m/s, radius of Earth at latitude of Glasgow $R = 6363.18 \times 10^3$m, and the nominal value for the latitude of Glasgow is 55.862° or 0.9749 rad. It is important to note that according to the WGS-84 local terrestrial gravity model these very small fluctuations in local acceleration due to gravity can be used in a back-calculation to obtain a corresponding fluctuation of $\pm 0.003514\%$ in the co-latitude around a nominal value of $\theta_{nom} = 0.597610$ rad/s. Whilst this is a very small absolute fluctuation it has a large relative effect on the LT signal itself. This is explained in detail in [3] and is shown to be of considerable significance for the signal processing. This aspect will be fully explored in a forthcoming publication.

2. Experiment scaling

2.1 Time scaling

The experiment is based on three distinct time scales. The first is the time scale of the swinging motion of the pendulum, of period $T_d = 4.1872$ s. The second is the rate of Foucault precession at the experimental location of Glasgow, which means that the pendulum will precess round through 360° in 28.917 hours. The third time scale relates to the time required to accrue the LT precession signal itself, noting that this is predicted to be at a rate of 162.6 mas/year. It is expected that the instrumentation may be capable of discriminating a signal of around 250 mas/year from the data and noise floor, notwithstanding the issues associated with co-latitude fluctuation, leading to an extraction time of ~ 1.55 years. The time scaling analysis can be portrayed on a bar chart in which the y-axis scaling values are shown on logarithmic scales, Fig. 6(a).

2.2 Length scaling

In addition to the time scaling this experiment also has interesting length scale characteristics. The distance traversed by the pendulum bob over one swing cycle is $4a$, (where a is the peak swing amplitude) hence 64 cm. The next scale length corresponds to one full rotation of the Foucault precession, which is $2\pi a = 100.5$ cm travelled by the bob over one full Foucault precession cycle of 360°. The third length scale relates to the circumferential distance travelled by the pendulum bob over the LT measurement period, and this is obtained by calculating the number of full precessions over the accrual time, therefore: 1.55 years / 28.917 hours = 469.5 precessions, which gives a total circumferential distance travelled by the bob of $2\pi a * 469.5 = 47199.3$ cm. The length scaling analysis is shown in Fig. 6(b).

The two scaling analyses are summarised below in the bar charts of Fig. 6. In Fig. 6(a) **T** denotes the period of one swing of the pendulum, **P** represents the time taken for one full 360° rotation of Foucault precession, and **L** gives the time for the accrual of 250 mas/year of LT precession. In Fig. 6(b) **R** defines the swing displacement of the bob over one full cycle, **F** denotes the circumferential distance for one Foucault precession cycle, and **M** gives the circumferential distance travelled over the accrual time. The time scales are well separated, by 4 and 2 decades respectively. The first two length scales are less clearly separated, but with over 2 decades of spacing between **F** and **M**. The accuracy of the underlying performance of the pendulum as a detector of Foucault precession cannot be overstated.

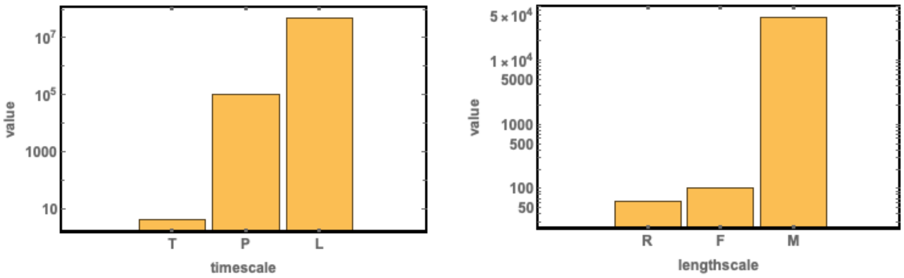


Fig. 6. (a) The three time scales (see text) shown on a $\log_{10}$ scale (units of s). (b) The three length scales (see text) also shown on a $\log_{10}$ scale (units of cm).

It is this behaviour that provides the steady-state periodic motion against which the small precessional motion due to frame-dragging might then be observed. So, in practice the maintenance of accurate and dependable **T** and **P** time scales is of the utmost importance, and this then guarantees maintenance of the **R** and **F** length scales. There are several factors that control this. The gap between the bottom face of the bob and the top of the coil pack is defined by $l_{gap} = 6 \pm 0.1$ mm, and the characteristic that then defines the relationship, shown in Fig. 5(b), between the amplitude of swing and the delay time between detection of the bob and the electromagnetic pulse that pushes it on in its swing is based on the preservation of that gap. l_{gap} is set initially with a slip gauge and then regularly measured indirectly using a pair of lasers which are set to pick up the relative vertical position of the bob. The elongation characteristic of the pendulum fibre was shown in [5] to be 0.69% over an initialisation period which then reduced to zero after 600 hours, at room temperature. This measurement did not take the relative humidity of the laboratory into account but in the five weeks of recent continuous testing the laboratory the room temperature, although controlled, has fluctuated over a range from 15°C up to 20.5°C, and the relative humidity has also been measured from a maximum of 51% down to a minimum of 44%, and yet zero change has been recorded in l_{gap} during that time. Other environmental effects that could potentially affect l_{gap} are seismic vibrations in the laboratory (minimised by the vibration isolation sub-system fitted) [5], aerodynamic currents, and electromagnetic compatability (EMC) problems. The laboratory is a nominally closed volume of space, only opened twice daily for ingress and egress, and EMC issues have been reduced as far as possible by the use of thermal and electrical screening [5]. Further improvements are planned to the thermal integrity of the laboratory space before the frame-dragging measurement is undertaken.

A still image from the recent testing programme is shown below in Fig. 7. The bob height sensing lasers can be seen and a 360° scale has been attached to the Eddy current damper ring. The physical location of the Eddy current damper ring was arrived at after extensive height adjustment tests prior to the commissioning test, to ensure that the balance between passive reduction in the ellipticity precision, in conjunction with the active element provided by the closed loop electromagnetic pusher system, did not reduce the swing amplitude below the set peak value of 16 cm, but did reduce the ellipticity precession to a minor axis peak amplitude of no more than 1 cm. This performance was achieved by setting the Eddy current damper ring so that its top surface is approximately 8 mm below the top surface of the coil pack. The levelling of both the coil pack and the Eddy current damper ring was found to be critical for consistent Foucault precession over time, and a precision digital leveling device was used to get the angular orientation of these components to within ±0.1°. This was measured along the axis of the optical bench, and at orientations of 45°, 90°, and 135°, to guarantee levelling of these two key components for the whole 360° of the Foucault precession. The coil pack orientation was adjusted by means of four linear and two angular micrometer adjusters fitted to the lower structure supporting each end of the grey optical bench beam (Fig. 7). The Eddy current damper ring orientation was adjusted by carefully setting four M10 nylon support bolts in their threaded nylon spacers, locating the ring accurately on the optical bench beam (this part of the installation is not visible in Fig. 7 because it is obscured in this view by the Eddy current damper ring).

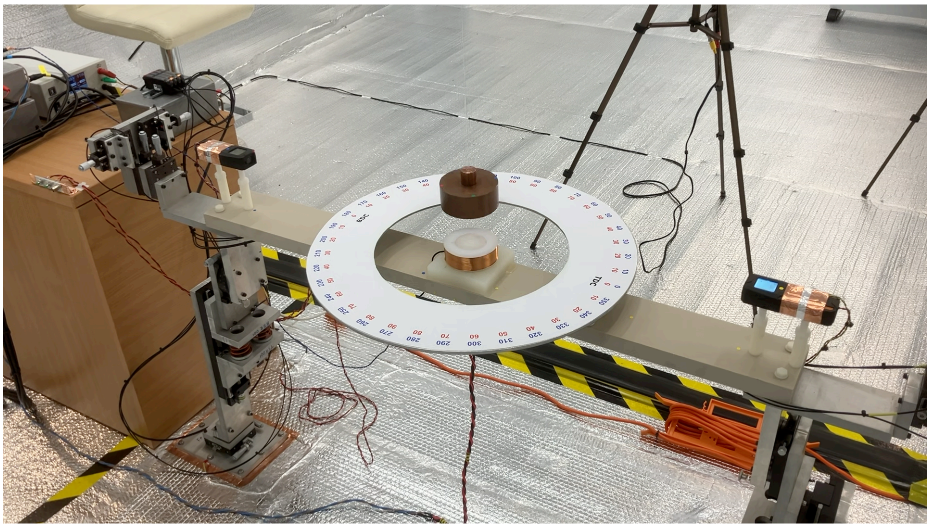


Fig. 7. Pendulum shown during the five-week commissioning test. Cables to the drive coil (central heavy flexible red and black) are a twisted pair, cables to the sense coil (light flexible red and black, to the left) also a twisted pair. Lasers for height sensing of the bob shown to left and right, also with twisted pair wiring. 360° degree paper disk fitted to the Eddy current damper ring for general observation of Foucault precession over time.

Conclusions

This paper summarises the final design of a short Foucault pendulum for continuous running in the laboratory. This pendulum is 4.359 m in length, and in comparison with Foucault's original pendulum of length 67 m this is a very short example. It is well known, and has been shown numerically in previous work, that short Foucault pendulums are prone to severe problems of ellipticity precision over short to medium time scales. This effect can be reduced by the combined use of passive damping through Eddy current effects and active control by the use of closed loop electromagnetic forcing. In this case the pendulum has both these features and has passed a five-week commissioning test of continuous running with no degeneration of the motion observed. This is not to say that there is no minor axis displacement, but that this is readily limited to <1 cm and then held constant over the long term, against a major axis swing amplitude typically of 10 to 25 cm, in this case 16 cm. The swing amplitude is controlled electronically by a combination of the peak drive current and the delay time, defined as the time interval between the receipt of the sense pulse and the firing of the drive pulse. The drive current is approximately 7A in this system, and both the sense and drive coils were wound professionally by a specialist manufacturer of high-performance inductors. The on-time of the drive pulse is also controllable but of less significance than the delay time. The mechanical design of the pendulum has been significantly improved in the working pendulum reported here, through the replacement of the upper pivot bearing with a simple but geometrically precise fibre attachment sub-assembly, removal of all ferrous material within the bob, modifications to the geometry of the bob to move the natural wobble frequencies well away from any of the operating frequencies, and the selection of a neodymium bob magnet with the most uniform magnetic flux density available. Angular alignment of all the critical parts of the installation has been implemented to $\pm 0.1°$. The timescales associated with the swing of the pendulum and the Foucault precession of the bob are shown to be of principal importance for long term accuracy, and maintenance of these motions is achieved over long time periods by means of the closed loop electromagnetic drive system and the Eddy current damper ring. A critically sensitive metric for the maintenance of the pendulum's swing response is the gap between the lower face of the bob and neodymium magnet, and the upper face of the concentric coil pack. This is denoted here by l_{gap} and it is initially set using an accurate slip gauge, and then monitored in this installation by means of two lasers. The key to maintaining this gap in the long term is the use of a fibre material with minimised room-temperature creep and elongation, close control of the laboratory's environmental temperature and relative humidity, the reduction of stray air currents, isolation from seismic vibrations, and the minimisation of EMC problems. Some attention has already been paid to these issues, but improvements to the laboratory temperature and relative humidity control are planned for the very near future. The pendulum as summarised in this paper has been run continuously for a five-week commissioning test and has performed faultlessly. This does not mean that it is immediately ready for use as an instrument for measuring the minute physical effect of frame-dragging due to the proximity and rotation of the Earth, but confirms that the experiment is moving in the right direction to attempt to meet that challenge in the next year.

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